M-ary Symbol Error Outage Over Nakagami-m Fading Channels in Shadowing Environments

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Abstract

This letter addresses the problem of finding a tractable expression for the symbol error outage (SEO) in flat Nakagami-m fading and shadowing channels. We deal with M-ary phase shift keying (M-PSK) and quadrature amplitude modulation (M-QAM) which extends our previous results on BPSK signaling. We propose a new tight approximation of the symbol error probability (SEP) holding for M-PSK and M-QAM signals which is accurate over all signal to noise ratios (SNRs) of interest. We derive a new generic expression for the inverse SEP which facilitates the derivation of a tight approximation of the SEO in a lognormal shadowing environment.

Index Terms

Mobile Communications, Symbol Error Probability, Symbol Error Outage, Shadowing, Fast Fading Channel, M-PSK, M-QAM.

I. Introduction

The offered quality of service (QoS) in wireless communication systems strongly depends on the channel conditions, impaired by fading, shadowing and pathloss.

As for fading, signal processing techniques are available to face the rapid variations of the instantaneous signal to noise ratio (SNR) due to multipaths. One of the most significant performance criterion in this context is the average symbol error probability (SEP) which is obtained by averaging over all fading channel states [1, 2].

Practical wireless applications such as cellular networks, however, not only suffer from fading but also from lognormally distributed shadowing which is a slowly varying process compared to fading [3]. Over a typical communication duration, the shadowing channel is non-ergodic because the average SNR does not pass through all statistical states, and the average SEP is hence not meaningful. Outage behavior is therefore typically invoked and the symbol error outage (SEO) quantifies the probability that a certain average SEP cannot be supported for a given shadowing statistics.

In order to obtain the SEO, we need to inverse the SEP w.r.t. the SNR. This task is not trivial and there has been minimal research efforts on this subject. Conti *et al.* have proposed invertible tight upper and lower bounds of the SEP for M-PSK signals with diversity reception in [4] facilitating a closed form expression of the SEO in shadowing environments. The authors extended their approach to find invertible bounds of Bit Error Probability (BEP) for M-QAM signals but they limit themselves to Rayleigh fading channels [5]. In [6], the authors extend their previous results to Nakagami-m fading channels. In [7], we proposed a new approach to find an invertible expression to the SEP based on the approximation of a hypergeometric function. We dealt with Nakagami-m fading channels but limited to BPSK signals.

The extension of the previous analysis to M-PSK and M-QAM signals over Nakagami-m fading channels in a lognormal shadowing environments is not trivial and will be exposed here. The paper is organized as follows. In Section II, we give a novel tight approximation to the known exact M-PSK/QAM SEP which holds in both non-asymptotic as well as asymptotic regimes. In Section III, we propose a closed form expression for the invertible SEP w.r.t. the SNR for M-ary modulated signals. In Section IV, we then apply the inverted expression of the SEP to obtain the SEO in closed form. Conclusions are finally drawn in Section V.

II. New Tight Approximations of the SEP

A. *Exact SEP over Nakagami-m Channels*

We consider linearly modulated signals, for which the probability of a symbol error event E, i.e. the SEP, for M-PSK signals over a flat Nakagami-m fading channel with average SNR $\overline{\gamma}_s$ is given by [1, 8]:

$$\begin{aligned} P_s(E|\overline{\gamma}_s) &= M_{\gamma_s}(-g_{psk})\left\{\frac{1}{2\sqrt{\pi}}\frac{\Gamma(m+1/2)}{\Gamma(m+1)}{}_2F_1\left(m,\frac{1}{2};m+1;\frac{1}{1+g_{psk}\overline{\gamma}_s/m}\right)\right. \\ &\quad \left.+\frac{\sqrt{1-g_{psk}}}{\pi}F_1\left(\frac{1}{2},m,\frac{1}{2}-m;\frac{3}{2};\frac{1-g_{psk}}{1+g_{psk}\overline{\gamma}_s/m},1-g_{psk}\right)\right\}, \end{aligned} \tag{1}$$

and for M-QAM signals as:

$$\begin{aligned} P_s(E|\overline{\gamma}_s) &= \frac{2gM_{\gamma_s}(-g_{qam})}{\sqrt{\pi}}\frac{\Gamma(m+1/2)}{\Gamma(m+1)}{}_2F_1\left(m,\frac{1}{2};m+1;\frac{1}{1+g_{qam}\overline{\gamma}_s/m}\right) \\ &\quad -\frac{g^2}{\pi}\frac{\Gamma(m+1/2)}{\Gamma(m+3/2)}M_{\gamma_s}(-2g_{qam})F_1\left(1,m,1;m+\frac{3}{2};\frac{1+\frac{g_{qam}\overline{\gamma}_s}{m}}{1+\frac{2g_{qam}\overline{\gamma}_s}{m}},\frac{1}{2}\right). \end{aligned} \tag{2}$$

Here, ${}_2F_1(a,b;c;u)$ is the Gauss hypergeometric function with 2 parameters of type 1 and 1 parameter of type 2 [9, §9.14.1)]. The function $F_1(a,b,b';c;v,w)$ is the Appell hypergeometric function of the first kind [9]. Moreover, $\Gamma(.)$ is the Gamma function, $M_{\gamma_s}(-s)=(1+s\overline{\gamma}_s/m)^{-m}$ is the moment generating function of the SNR γ_s in a Nakagami-m fading channel, $g_{psk}=\sin^2(\pi/M)$, $g_{qam}=3/(2(M-1))$ and $g=1-1/\sqrt{M}$ are modulation dependent constants.

B. Tight Approximation of SEP

The exact expressions of the SEP for M-PSK and M-QAM given by (1) and (2) are not invertible. We present our first fundamental result on the approximation of the SEP for M-PSK and M-QAM signals in the following proposition:

Proposition 1 (SEP approximation) *Over non-frequency selective Nakagami-m fading channels with an average SNR $\overline{\gamma}_s$ high enough, the average SEP is well approximated by:*

$$P_s(E|\overline{\gamma}_s) \approx k_{mod} \frac{x^m}{\sqrt{1 - x\tilde{t}}}; \forall \overline{\gamma}_s \tag{3}$$

with $\tilde{t} = m/(m+1)$, $x = 1/\left(1 + g_{mod}\overline{\gamma}_s/m\right)$, g_{mod} being equal to g_{psk} or g_{qam}, and k_{mod} being equal to:

$$\begin{aligned} k_{psk} &= \frac{\Gamma(m+1/2)}{2\sqrt{\pi}\Gamma(m+1)} + \frac{\sqrt{1-g_{psk}}}{\pi} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - m; \frac{3}{2}; 1 - g_{psk}\right), && (4) \\ k_{qam} &= \frac{2g\Gamma(m+1/2)}{\sqrt{\pi}\Gamma(m+1)} - \frac{g^2\Gamma(m+1/2)}{2^m\pi\Gamma(m+3/2)} {}_2F_1\left(1, m+1; m+\frac{3}{2}; \frac{1}{2}\right), && (5) \end{aligned}$$

for M-PSK and M-QAM signals, respectively.

Proof: We start with M-PSK signals. The Gauss hypergeometric function in (1) can be approximated in a very efficient manner thanks to the Laplace approximation [10]. In the asymptotic regime, the Gauss hypergeometric function is well approximated by $1/\sqrt{1 - x\tilde{t}}$ [7]. Moreover, in the asymptotic regime, the Appell hypergeometric function converges to a constant Gauss hypergeometric function [9]:

$$\lim_{\overline{\gamma}_s \to \infty} F_1\left(\frac{1}{2}, m, \frac{1}{2} - m; \frac{3}{2}; \frac{1 - g_{psk}}{1 + g_{psk}\overline{\gamma}_s/m}, 1 - g_{psk}\right) = {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - m; \frac{3}{2}; 1 - g_{psk}\right). \tag{6}$$

Noticing that $M_{\gamma_s}\left(-g_{psk}\right) = x^m$, we can hence write the high SNR expression of the SEP as:

$$P_s(E|\overline{\gamma}_s \to \infty) \approx \frac{x^m}{\sqrt{1 - x\tilde{t}}} \left\{ \frac{\Gamma(m+1/2)}{2\sqrt{\pi}\Gamma(m+1)} + \frac{\sqrt{z}}{\pi} {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - m; \frac{3}{2}; z\right) \sqrt{1 - x\tilde{t}} \right\}, \tag{7}$$

with $z = 1 - g_{psk}$. Moreover, $\lim_{\overline{\gamma}_s \to \infty} \sqrt{1 - x\tilde{t}} = 1$, which completes the proof for M-PSK signals.

As for M-QAM signals, let us redefine $x = 1/\left(1 + g_{qam}\overline{\gamma}_s/m\right)$ and $x' = 1/\left(1 + 2g_{qam}\overline{\gamma}_s/m\right)$. We remark that $M_{\gamma_s}\left(-g_{qam}\right) = x^m$ and $M_{\gamma_s}\left(-2g_{qam}\right) = \left(x'\right)^m$. Hence, the expression in (2) can be rewritten as:

$$P_s(E|\overline{\gamma}_s) = x^m \left(\frac{2g}{\sqrt{\pi}}\frac{\Gamma(m+1/2)}{\Gamma(m+1)}{}_2F_1\left(m,\frac{1}{2};m+1;x\right) - \left(\frac{x'}{x}\right)^m \frac{g^2}{\pi}\frac{\Gamma\left(m+1/2\right)}{\Gamma\left(m+3/2\right)}F_1\left(1,m,1;m+\frac{3}{2};\frac{x'}{x},\frac{1}{2}\right)\right). \tag{8}$$

The Gauss hypergeometric function is again approximated by the Laplace method which, in asymptotic regime, yields $1/\sqrt{1-x\tilde{t}}$. Moreover, $\lim_{\overline{\gamma}_s\to\infty} x'/x = 1/2$ and hence the Appell hypergeometric function converges towards a constant Gauss hypergeometric function [9]:

$$\lim_{\overline{\gamma}_s\to\infty} F_1\left(1,m,1;m+\frac{3}{2};\frac{x'}{x},\frac{1}{2}\right) = {}_2F_1\left(1,m+1;m+\frac{3}{2};\frac{1}{2}\right). \tag{9}$$

By using the asymptotic expression of the Appell hypergeometric function in (8), the high SNR expression of the SEP for M-QAM signals is:

$$P_s(E|\overline{\gamma}_s\to\infty) \approx \frac{x^m}{\sqrt{1-x\tilde{t}}}\left(\frac{2g\Gamma(m+1/2)}{\sqrt{\pi}\Gamma(m+1)} - \frac{g^2\Gamma(m+1/2)}{\pi\Gamma(m+3/2)}\times {}_2F_1\left(1,m+1;m+\frac{3}{2};\frac{1}{2}\right)\left(\frac{x'}{x}\right)^m\sqrt{1-x\tilde{t}}\right). \tag{10}$$

In asymptotic regime, we have $\sqrt{1-x\tilde{t}}\to 1$, which completes the proof for M-QAM signals.

For both M-PSK and M-QAM signals, the proof has been derived in asymptotic regime. However, the use of the hypergeometric function allowed to achieve very tight approximations, even at relatively low SNR when compared to usual approximations. Indeed, the average relative error is found to be less than 1.70% in all the range of SNRs of interest, i.e. where the SEP is between 10^{-1} and 10^{-4} for uncoded systems. As can be inferred from Figure 1, our new approximation is really close to the exact values of the SEP for three modulations (QPSK, 16-QAM and 64-QAM) and three values of channel severity $m = 1, 3, 6$.

III. Inversion of SEP

From Proposition 1, it could be possible to find a simple lower bound of the SEP by replacing the denominator of (3) by $\sqrt{1-x^m\tilde{t}}$ as proposed in [7] for BPSK signals. This lower bound would allow to invert the average SEP w.r.t. the SNR. However, it results in a poor accuracy for the SNR range of interest, especially for large m. We hence propose a more accurate inversion in the following proposition:

Proposition 2 (SEP inversion) *Over non-frequency selective Nakagami-m fading channels, the closed form expression of the SNR versus the average SEP is well approximated by:*

$$\overline{\gamma}_s(P_s^*(E)) = c_0\left(P_s^*\left(E\right)^{-\frac{1}{m}}\left(1 - c_1 P_s^*\left(E\right)^{\frac{1}{m}}\right)^{-\frac{1}{2m}} - \left(k_{mod}\right)^{-\frac{1}{m}}\right), \tag{11}$$

with $c_0 = m\sqrt[m]{k_{mod}}/g_{mod}$*,* $c_1 = \tilde{t}/\sqrt[m]{k_{mod}}$ *and* k_{mod}*,* g_{mod} *as defined in Proposition 1.*

Proof: Squaring (3), the inverse problem is reduced to solving the following equation in $x \in [0,1]$:

$$P(x) = \left(k_{mod}\right)^2 x^{2m} + \left(P_s^*(E)\right)^2 \tilde{t}x - \left(P_s^*(E)\right)^2 = 0, \tag{12}$$

with $P_s^*\left(E\right)$ being the average target SEP. This equation does not have solutions in closed form for any values of m. We therefore solve the equation by an iterative method in order to approach the solution x_s of (12), which is initialized by removing the middle term. The solution of the new equation is then $x_1 = \sqrt[m]{P_s^*\left(E\right)/k_{mod}}$. We define the following series $\left(x_n\right)_{n\geq 0}$ built by replacing the middle term of (12) by the new estimation of the solution:

$$\begin{cases} x_0 &= 0, \\ x_{n+1} &= \left(\dfrac{P_s^*\left(E\right)}{k_{mod}}\right)^{\frac{1}{m}}\left(1 - \tilde{t}x_n\right)^{\frac{1}{2m}}. \end{cases} \tag{13}$$

Let us define the functional mapping f as: $f : [0,1] \to [0,1]$ such as $f(x) = \left(\frac{P_s^*(E)}{k_{mod}}\right)^{\frac{1}{m}}\left(1 - \tilde{t}x\right)^{\frac{1}{2m}}$. The function f is continuous and differentiable in $[0,1]$ and one can verify that f is a monotonically decreasing function in $[0,1]$. Moreover, $\forall x \in [0,1]$:

$$\left|f'\left(x\right)\right| = \frac{\tilde{t}}{2m}\left(\frac{P_s^*\left(E\right)}{k_{mod}}\right)^{\frac{1}{m}}\frac{1}{\left(1 - \tilde{t}x\right)^{\frac{2m-1}{2m}}}. \tag{14}$$

According to the above expression, we can verify that $\exists\rho \in]0,1[$ as $\forall x \in [0,1]$, we have $\left|f'\left(x\right)\right| \leq \rho = \left|f'\left(1\right)\right| < 1$ [1]. Hence, the application f is a contracting mapping with a ρ-ratio. Invoking the fixed-point theorem [11], $f\left(x\right) = x$ as a unique solution in $[0,1]$. Since $P\left(x\right) = 0 \Leftrightarrow f(x) = x$, the unique fixed point of f, x_s, is the solution of (12) and all series obeying $x_{n+1} = f\left(x_n\right)$ are converging towards x_s. In other words, $\lim_{n\to\infty}\left(x_n\right) = x_s$ and the

[1] This condition only holds if $P_s^*\left(E\right) < 2^m k_{mod}\sqrt{m+1}$, where k_{mod} is a constant depending on m and the modulation order M. However, for the considered values of $P_s^*\left(E\right)$, i.e. $P_s^*\left(E\right) \leq 10^{-1}$, the condition holds.

proof is complete. We stop this sequence at the second iteration, i.e. $x_2 = f(f(x_0))$, and we obtain the SNR $\overline{\gamma}_s$.

In Figure 2, we plot the relative error between the exact solution of (12) and the approximate solution of this equation, obtained from Proposition 2 (remember that $x = 1/(1 + g_{mod}\overline{\gamma}_s/m)$). The relative error in percent is plotted versus the target SEP $P_s^*(E)$ and labeled on the fading severity parameter m and the modulation order M. The target SEP is ranging from 10^{-4} to 10^{-1} which is the SEP range of interest without coding. We observe that the smaller the target SEP, the smaller the difference between the exact and approximated solutions. The worst precision is reached for a QPSK signal in a low fading severity channel (i.e. $m = 6$) for a target SEP about 10^{-1}. The relative error is less than 2%. The relative error increases as the parameter m increases, and it decreases as the modulation order M increases. We observe that for 64-QAM signals, the relative error is less than 1.4% for $m = 6$ and $P_s^*(E) = 10^{-1}$.

IV. Symbol Error Outage Approximation

Over the duration of a wireless communication, the average SNR per symbol $\overline{\gamma}_s$ varies due to shadowing. The average SEP can hence not be supported by the channel over the duration of an application connection, which is why the outage of the SEP, i.e. the SEO, is invoked. This is similar to the Bit Error Outage (BEO) dealt with in [7,12]. The SEO is defined as the probability that the SEP exceeds a given threshold, P_s^* [4]:

$$P_s(O) = Pr(P_s(E|\overline{\gamma}_s) > P_s^*), \tag{15}$$

which is easily shown to be equivalent to

$$P_s(O) = \int_0^{\overline{\gamma}_s(P_s^*)} p_{\overline{\gamma}_s}(\xi)d\xi, \tag{16}$$

where $\overline{\gamma}_s(P_s^*)$ is the required SNR to reach the target SEP P_s^* and $p_{\overline{\gamma}_s}(\overline{\gamma}_s)$ is the probability density function (pdf) of $\overline{\gamma}_s$.

We consider that $\overline{\gamma}_s$ is lognormally distributed with parameters μ_{dB} and σ_{dB} (i.e. the mean and the standard deviation of SNR considering shadowing expressed in dB), which allows the SEO to be derived in closed form as [12]:

$$P_s(O) = Q\left(\frac{\mu_{dB} - 10\log_{10}\overline{\gamma}_s(P_s^*)}{\sigma_{dB}}\right), \tag{17}$$

where $Q(x)$ is the Gaussian Q-Function. Thanks to (11), an approximation of the SEO can be easily derived.

To verify the proposed approximations, Figure 3 provides the SEO for a target SEP of $P_s^* = 10^{-2}$ versus the shadowing mean μ_{dB} for $\sigma_{dB} = 8$ (which is a typical value for outdoor environments) and labeled on the fading factor m for QPSK, 16-QAM and 64-QAM. Our approximation is compared to the exact SEO values obtained by numerically inverting the exact SEP expressions given in (1) and (2). The proposed approximation of SEO is observed to be very tight for all values of μ_{dB} with the error never exceeding 1.90%. We have also plotted the SEO versus the shadowing standard deviation σ_{dB} for $\mu_{dB} = 30$ in Figure 4. The average relative error does not exceed 3.25% in this case. We can compare the precision of our findings to the bounds of the SEO presented in [4]. Although the authors gave SEO for M-PSK signals and a multi-branch receiver over a Rayleigh fading channel, we can apply in a straightforward manner their results to SISO systems in Nakagami-m fading channels. Indeed SISO Nakagami-m acts as a SIMO N branches Rayleigh channel with $N = m$ and dividing the SNR by m. For a QPSK signal as an example, a target SEP $P_s^* = 10^{-2}$ and a lognormal standard deviation of $\sigma_{dB} = 8$, the accuracy level of the tightest bound presented in [4] is about 0.0032% for $m = 1$ (i.e. Rayleigh Channel), and 0.11% with our approximation. The accuracy of the tightest bound in [4] for $m = 6$ is about 4.60% whereas with our approximation about 1.90%. Hence our SEO approximation is more accurate than the ones presented in [4] when the fading parameter m increases.

V. Conclusions

In this letter, we presented a new tight approximation of the SEP for M-PSK as well as M-QAM signals experiencing a non-frequency selective Nakagami-m fading channel which extends our previous results on BPSK signaling in a not straightforward manner. The new approximation is accurate for all SNR values of interest, i.e. where the SEP is between 10^{-1} and 10^{-4}. The latter proved to be invertible w.r.t. the average fading gain. This facilitated a tight approximation of the SEO to be obtained in closed form for a given lognormally distributed shadowing environment. Obtained insights can be used for resource allocation protocols, network planning, and connectivity analysis.

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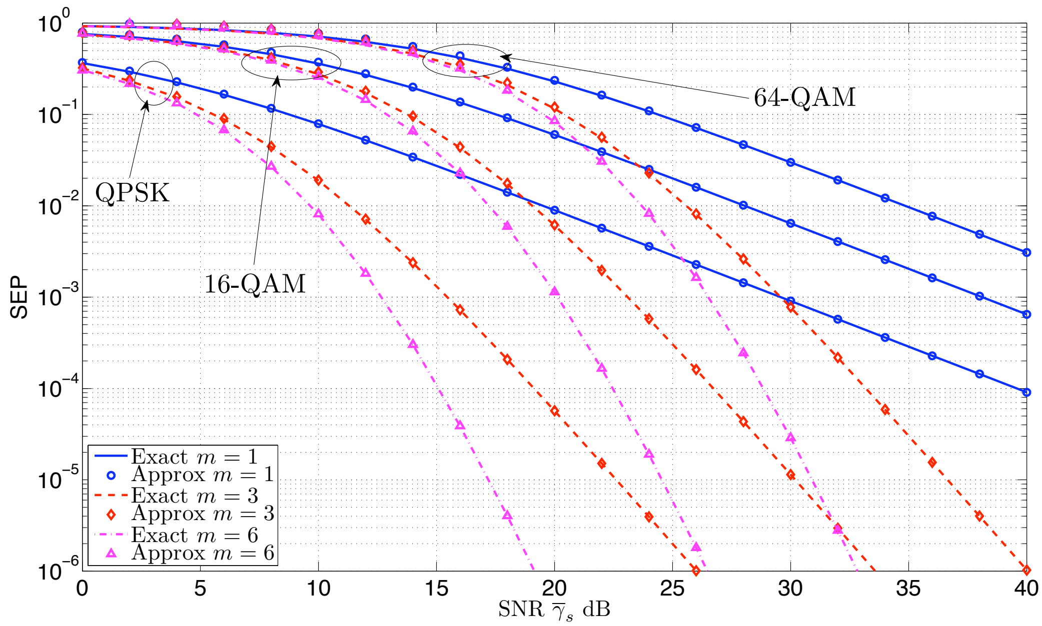

Fig. 1

SEP of QPSK, 16-QAM and 64-QAM signals over Nakagami-m channels using the exact expressions in (1), (2) and its approximation in (3).

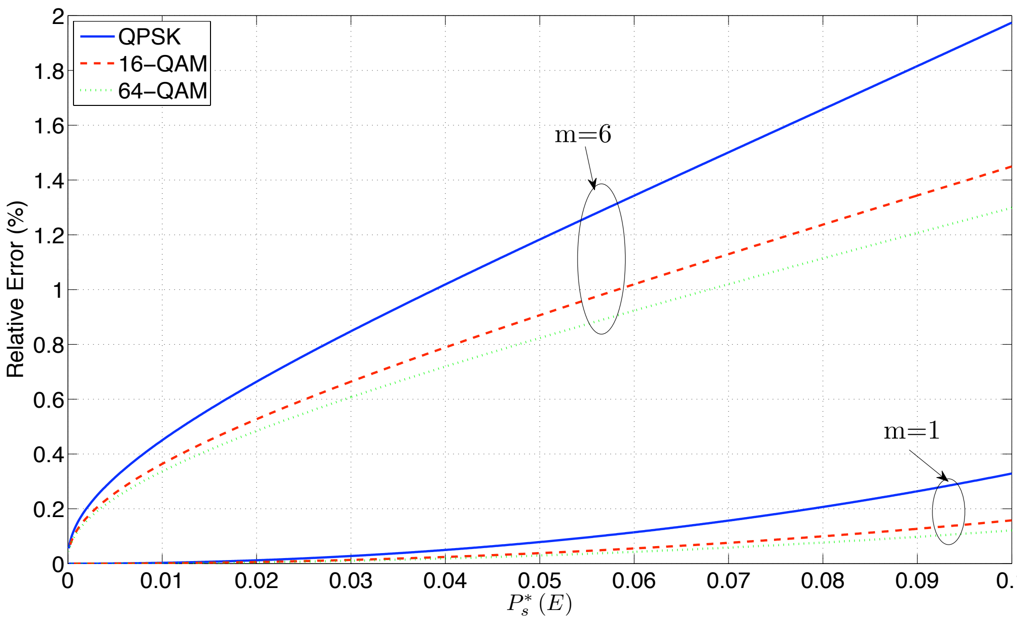

Fig. 2

RELATIVE ERROR IN PERCENT BETWEEN THE EXACT SOLUTION OF (12) AND THE APPROXIMATION OF THIS SOLUTION OBTAINED IN THE PROOF OF THE PROPOSITION 2.

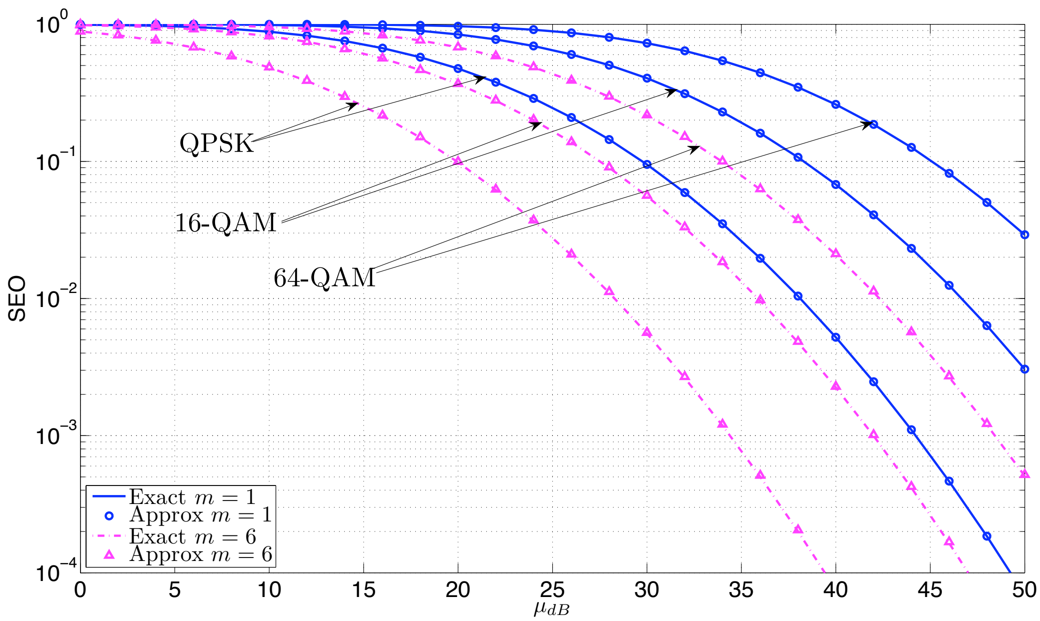

Fig. 3

SEO of QPSK, 16-QAM and 64-QAM over Nakagami-m fading channels numerically inverting the exact expressions in (1) and (2) and using the closed form expression in (11) and (17), assuming $\sigma_{dB} = 8$ and $P_s^* = 10^{-2}$.

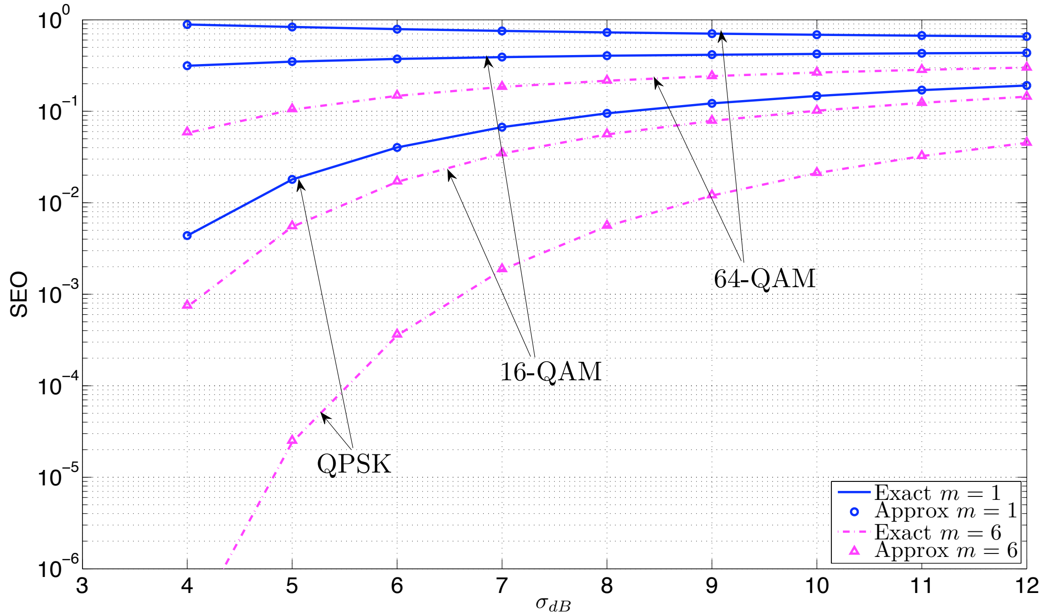

Fig. 4

SEO OF QPSK, 16-QAM AND 64-QAM OVER NAKAGAMI-m FADING CHANNELS NUMERICALLY INVERTING THE EXACT EXPRESSIONS IN (1) AND (2) AND USING THE CLOSED FORM EXPRESSION IN (11) AND (17), ASSUMING $\mu_{dB} = 30$ AND $P_s^* = 10^{-2}$.